Latin Squares: Diagonal Complete Latin Squares and Complete Latin Cubes
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1 Diagonal Complete Latin Squares

(Author: Jenny Zhang)
Definition: Right-diagonal complete latin square:

We call a Latin Square A=(a_{ij}) of order n, $a_{ij} \in \{1, 2, .., n\}$, and which i stands for row and j stands for column, right-diagonal complete latin square if each ordered pair$\{(a_{ij}, a_{i+1,j+1}); 1 \leq i, j \leq n\}$ only shows up once in a latin square.

Example: Right-diagonal complete latin square of order 9

1	2	3	4	5	6	7	8	9
2	6	1	5	9	4	8	3	7
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
9	7	5	3	1	8	6	4	2
3	1	8	6	4	2	9	7	5
5	9	4	8	3	7	2	6	1
8	3	7	2	6	1	5	9	4
6	4	2	9	7	5	3	1	8

This is a right-diagonal latin square but not a left-diagonal latin square. For example, $(A_{19}, A_{28}) = (9, 3)$, $(A_{88}, A_{79}) = (9, 3)$

Definition: Left-diagonal complete latin square:

The definition of left-diagonal latin square is similar with right-diagonal latin square. It's when the ordered pair$\{(a_ij, a_{i+1,j-1}) : 1 \leq i, j \leq n\}$only shows up once in a latin square, then it's left-diagonal latin square.

Diagonal Complete Latin Square

We call a Latin Square diagonally complete latin square is when that latin square is both right-diagonal complete and left-diagonal complete.

Why do diagonal complete latin square exist?

The total collection of all possible pairs in a Latin square are n^2 because we have n symbols, and each symbol has n other possibilities to be paired with, so we have $n \cdot n$ possible pairs, or n^2.

The total number of diagonal pairs that are in a latin square is $(n-1)^2$. For a $n*n$ latin square, in the right direction, there is no number pair with the numbers in the last row. So there are only $(n-1)^2$ pairs of number in one direction. Same as in the left direction, there is no number pair with the numbers in the first row.

Examples: Diagonal Complete Latin Square
n=2:

1	2
2	1

n=3:
We can start by fill the first row 1,2,3 and without loss of generality because we can name these symbols anything to make it 1, 2, 3. The two possible latin squares are:

1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

And those are not diagonal complete latin squares. So there is no diagonal complete latin squares for order 3.
When n=4:

1	2	3	4
2	1	4	3
4	3	2	1
3	4	1	2

We have been trying to make diagonal complete latin squares for orders 5 and 6 but have not found one yet. But we know that one exists for order 9, which is shown above.

2 Complete Latin Cubes

(Author: Tamara Gomez)
Another way to generalize complete Latin squares is to look at complete Latin cubes. First we need to understand what Latin cubes are.

Definition. A **Latin cube** L of order n is an $n \times n \times n$ cube with n^3 cells filled with n symbols (usually $\{1, 2 \ldots, n\}$) so that no symbol is repeated in any row, column, and aisle. Since a latin cube is $n \times n \times n$, a Latin cube consists of n Latin squares in each dimension, so it consists of $3n$ Latin squares total. Before we look at the definition of complete Latin cubes, we must talk about notation. Let $L_{i,j,k}$, where $1 \leq i, j, k \leq n$, be the entry of the cell in the ith row, jth column, and kth aisle of Latin cube L.

Example. This is a Latin cube of order 2. Since we are limited to two-dimensional paper, imagine that the two Latin squares are layered behind one another and form a 4×4×4 cube.

1	2
2	1

2	1
1	2

Definition. A **row complete** Latin cube L of order n is a Latin cube where each ordered pair $(L_{i,j,k}, L_{i,j+1,k})$, for some fixed k, where $1 \leq k \leq n$, only appears once, each ordered pair $(L_{i,j,k}, L_{i,j,k+1})$, for some fixed j, where $1 \leq j \leq n$, only appears once, and each ordered pair $(L_{i,j,k}, L_{i,j+1,k})$, for some fixed i, where $1 \leq i \leq n$, only appears once.

Definition. A **column complete** Latin cube L of order n is a Latin cube where each ordered pair $(L_{i,j,k}, L_{i+1,j,k})$, for some fixed k, where $1 \leq k \leq n$, only appears once, each ordered pair $(L_{i,j,k}, L_{i+1,j,k})$, for some fixed j, where $1 \leq j \leq n$, only appears once, and each ordered pair $(L_{i,j,k}, L_{i,j,k+1})$, for some fixed i, where $1 \leq i \leq n$, only appears once.

Definition. A **complete** Latin cube L of order n is a Latin cube that is both row complete and column complete.

Complete Latin Cube Construction.

The construction for Complete Latin cubes is very similar to the construction for complete Latin squares. Recall the construction for a Latin square of even order (we will be using the symbols $\{0, 1, \ldots, n-1\}$ rather than $\{1, 2, \ldots, n\}$ because we will be using modular arithmetic):

- Take an $n \times n$ Latin square L, where n is even.

- Let the entries of the first row be in the following order:

$$0, 1, n-1, 2, n-2, 3, n-3, \ldots, \frac{n}{2}+1, \frac{n}{2}$$

- Let the entries in row k be one greater than the entires in row $k-1$, where the arithmetic is done $\bmod n$. This will give us a row complete Latin square.

- Rearrange the order of the rows so that the first column is in the same order as the first row. I.e. the entries of the first column are in the following order as well:

$$0, 1, n-1, 2, n-2, 3, n-3, \ldots, \frac{n}{2}+1, \frac{n}{2}$$

  Now we have a complete Latin square.

It is important to remember this construction because we will use complete Latin squares to construct a complete Latin cube. Unfortunately we are limited to two-dimensional paper, and it is rather difficult to draw a three dimensional Latin cube. So, in order to visualize the Latin cube, we will be referring to "layers" to help us refer to depth, or the third dimension. Each latin cube has n "layers", where a layer is an $n \times n$ Latin square.

How to construct a complete Latin cube:

- Let the first layer be a complete Latin square that has been constructed using the method described above.

- Let the entries in layer k be one greater than the entries in layer $k-1$, where the arithmetic is done $\mod n$. I.e. The entries of the first row of our Latin square in the second layer will be in the order:

$$1, 2, 0, 3, n-1, 4, n-2, \ldots, \frac{n}{2}+2, \frac{n}{2}+1$$

  This will give us n complete Latin squares in one dimension, but we will only have either row complete or column complete in the other two dimensions.

- Rearrange the order of the layers, so that when we look at the upper left cell of each layer, those cells are in the order:

$$0, 1, n-1, 2, n-2, 3, n-3, \ldots, \frac{n}{2}+1, \frac{n}{2}$$

  This is the same method we used when we rearranged the rows of a row complete Latin square to create a complete Latin square. Now we have a complete Latin cube.

Example.
Here is a complete Latin cube L of order 4:
These are the layers of Latin squares from left to right:

Layer 1

0	1	3	2
1	2	0	3
3	0	2	1
2	3	1	0

Layer 2

1	2	0	3
2	3	1	0
0	1	3	4
3	0	2	1

Layer 3

3	0	2	1
0	1	3	2
2	3	1	0
1	2	0	3

Layer 4

2	3	1	0
3	0	2	1
1	2	0	3
0	1	3	2

These are the layers of Latin squares from top to bottom:

2	3	1	0
3	0	2	1
1	2	0	3
0	1	3	2

Layer 1

3	0	2	1
0	1	3	2
2	3	1	0
1	2	0	3

Layer 2

1	2	0	3
2	3	1	0
0	1	3	4
3	0	2	1

Layer 3

0	1	3	2
1	2	0	3
3	0	2	1
2	3	1	0

Layer 4

These are the layers of Latin squares from front to back:

Layer 1

0	1	3	2
1	2	0	3
3	0	2	1
2	3	1	0

Layer 2

1	2	0	3
2	3	1	0
0	1	3	4
3	0	2	1

Layer 3

3	0	2	1
0	1	3	2
2	3	1	0
1	2	0	3

Layer 4

2	3	1	0
3	0	2	1
1	2	0	3
0	1	3	2

Notice that each dimension has the same four distinct Latin squares. Also notice that each distinct Latin square is a complete Latin square because each ordered pair only appears once in each row, and once in each column. Therefore L is a complete Latin cube. Since our method of construction for complete Latin cubes is based off of the construction for complete Latin squares, for every order that there exists a complete Latin square, there also exists a complete Latin cube. Next week, we will explore different ways of thinking about complete Latin cubes.

Sources: “Diagonal complete Latin squares” Olaff Kraft, Herbert Palings, & Martin Shaffer.
“Latin Cubes and Hypercubes” Australian National University.
“Latin Squares and Their Applications” J. Denes & A.D. Keedwell.